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ON THE PARTIAL DIFFERENCE EQUATIONS OF MATHEMATICAL PHYSICS

BY

R. COURANT
K. FRIEDRICHS
H. LEWY

A Translation from

Mathematische Annalen, Vol. 100, pp. 32-74, 1928

BY

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AND

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University of California, Los Angeles

Scientific Report No. 2

DYNAMICAL WEATHER PREDICTION PROJECT
Department of Meteorology
18 December 1959

This translation has been sponsored in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contracts No. AF 19 (604)-3886 and AF 19(604)-4965.

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TRANSLATOR'S PREFACE

As research in dynamical weather prediction proceeds, increasing attention is being given to the mathematical problems encountered in the computation of numerical solutions of the dynamical equations for the atmosphere; the meteorologist's attention is thus drawn to problems of numerical analysis. Basic to much of this work are the classical existence and uniqueness proofs of Courant, Friedrichs and Lewy for elliptic and hyperbolic partial difference equations. The present translation has been prepared in order to fill the need for an English version of this paper, and thereby to make this fundamental work more readily available and useful to the meteorological profession in particular. After this translation was completed, however, my attention was drawn to a previous English translation prepared by M. H. Rand in 1956 at the British Atomic Energy Research Establishment, Harwell. The comparison is in general reassuring, and a number of errors noted in Rand's translation have here been avoided.

Particular care has been taken in the present translation to preserve the spirit of the original German as closely as possible. In this connection Miss Oberländer, of the Department of Meteorology, U.C.L.A., has contributed significantly, as well as assisting in the preparation of the final manuscript. The figures have been redrafted and enlarged to improve their readability, although no other changes of format have been made.

W. Lawrence Gates
Project Director

ON THE PARTIAL DIFFERENCE EQUATIONS OF MATHEMATICAL PHYSICS

By

R. Courant, K. Friedrichs, and H. Lewy

In dealing with classical linear differential equations, if one replaces the differential quotients by difference quotients defined over some -- let us assume rectangular -- grid, one obtains algebraic problems of a much more transparent structure. The following paper will undertake an elementary discussion of these algebraic problems and will principally discuss the behavior of the solution as the mesh width of the grid goes to zero. To these ends we limit ourselves mainly to the simplest, but nevertheless typical, cases which we handle in such a manner that the applicability of our methods to more general difference equations with arbitrarily many independent variables will be clear.

Corresponding to the familiar problems of differential equations, we will treat boundary value and eigenvalue problems for elliptic difference equations, and the initial value problem for hyperbolic or parabolic difference equations. We shall prove with simple typical examples that the limiting process is always possible, i.e., that the solutions of the difference equations converge toward the solutions of the corresponding differential equations; indeed we shall find that for elliptic equations in general, the difference quotients of arbitrarily high order tend toward the corresponding differential quotients. The existence of a solution to the differential equation is nowhere assumed and indeed we obtain through the limiting process a simple existence proof.[1] But while for the elliptic case convergence is the rule independently of the choice of grid, we will show that in the case of the initial value problem for hyperbolic equations convergence can be assumed only when the ratio of grid mesh sizes in different directions satisfies certain inequalities, which in turn are determined by the position of the characteristics relative to the grid.

We take as a typical example of the elliptic case the boundary value problem of potential theory. The dependence of its solution on the solution of the corresponding difference equation has been, moreover, extensively treated during the past few years.[2] Of course, in our case, in contrast to the previous work, the particular special properties of the potential equation will not be explicitly used so that the application of our method to other problems is not overlooked.

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1 Our method of proof may be extended without difficulty to cover the boundary and eigenvalue problems for arbitrary linear elliptic differential equations, and the initial value problem for arbitrary linear hyperbolic differential equations.

2 J. le Roux, Sur le problème de Dirichlet, Journ. de mathém. pur. et appl., (6) 10 (1914), p. 189. R. G. D. Richardson, A New Method in Boundary Problems for Differential Equations, Trans. Amer. Math. Soc. 18 (1917), p. 489 ff. H. B. Phillips and N. Wiener, Nets and the Dirichlet Problem, Publ. of the Mass. Institute of Technology (1925). (Translator's note: see J. Math. Phys., 2:105-124 (1923)).

Unfortunately these papers were missed by the first of the present three authors (Courant) in the writing of his note "Zur Theorie der partiellen Differenzengleichungen", Gött. Nachr., 23, X. 1925, which the present work extends.

Compare further: L. Lusternik, "Über einige Anwendungen der direkten Methoden in der Variationsrechnung", Recuil de la Société Mathém. de Moscou, 1926. G. Bouligand, "Sur le problème de Dirichlet", Ann. de la soc. polon. de mathém., 4 Krakau, 1926.

For the use of difference expressions, and for related works, see R. Courant, "Über direkte Methoden in der Variationsrechnung", Math. Annalen, 97, p. 711 and the literature cited there.

In addition to the stated principal part of the work, we shall append an elementary algebraic discussion of the boundary value problem for the elliptic equations connected with the well-known problem of random walks arising in statistics.

I. THE ELLIPTIC CASE

§ 1. Preliminary Remarks

1. Definitions. In the plane with rectangular coordinates x, y we consider first of all a square grid of points of mesh width $h>0$, all points having the coordinates $x=nh$, $y=mh$,

$$n, m = 0, \pm 1, \pm 2, \cdots .$$

Now let G be a region of the plane bounded by a continuous closed curve that is free of double points. Then the corresponding grid region G_h - which is uniquely determined for sufficiently small mesh width - shall consist of all those grid points that lie in G and which can be joined to a fixed or prescribed grid point of G by a connected series of grid points. We denote as a connected series of grid points a sequence of points such that each point in turn is one of the four neighboring points of the following point. We denote as a boundary point of G_h a point whose four neighboring points do not all belong to G_h. All other points of G_h we call interior points.

We shall consider functions u, v, ... of position in the grid, i.e., functions which are defined only for grid points. We shall also denote them as $u(x, y)$, $v(x, y)$, ... For their forward and backward difference quotients we employ the following abbreviations:

$$\frac{1}{h}\left[u(x+h,y)-u(x,y)\right]=u_x\,,\quad \frac{1}{h}\left[u(x,y+h)-u(x,y)\right]=u_y\,,$$

$$\frac{1}{h}\left[u(x,y)-u(x-h,y)\right]=u_{\bar{x}}\,,\quad \frac{1}{h}\left[u(x,y)-u(x,y-h)\right]=u_{\bar{y}}\,.$$

Correspondingly we form the difference quotients of higher order, e.g.,

$$(u_x)_{\bar{x}}=u_{x\bar{x}}=u_{\bar{x}x}=\frac{1}{h^2}\left[u(x+h,y)-2u(x,y)+u(x-h,y)\right],\ \text{etc.}$$

2. <u>Difference Expressions and Green's Transformations</u>. We got the simplest general view of linear difference expressions of the second order from the pattern of the theory of partial differential equations if we form from two functions u and v and their forward difference quotients a bilinear expression

$$\begin{aligned} B(u,v) = {} & a u_x v_x + b u_x v_y + c u_y v_x + d u_y v_y \\ & + \alpha u_x v + \beta u_y v + \gamma u v_x + \delta u v_y + g u v, \end{aligned}$$

where

$$a=a(x,y),\ \cdots,\ \alpha=\alpha(x,y),\ \cdots,\ g=g(x,y)$$

are functions in the grid.

From the bilinear expression of the first order we derive a difference expression of the second order in the following way: we form the sum

$$h^2\sum_{G_h}\sum B(u,v)$$

over all points of a region G_h in the grid wherein $B(u,v)$ is set to zero for the difference quotient between a boundary point and a point that does not belong to G_h. We now transform the sum through partial summation (i.e., we arrange according to v), and split it into a sum over the set of interior points G_h' and a sum over the set of boundary points Γ_h. Thus we obtain

$$h^2 \sum\sum_{G_h} B(u,v) = -h^2 \sum\sum_{G_h'} v\,L(u) - h \sum_{\Gamma_h} v\,R(u). \quad (1)$$

$L(u)$ is the linear "difference expression of second order" defined for all interior points of G_h,

$$L(u) = (a u_x)_{\bar{x}} + (b u_x)_{\bar{y}} + (c u_y)_{\bar{x}} + (d u_y)_{\bar{y}}$$
$$- \alpha u_x - \beta u_y + (\gamma u)_{\bar{x}} + (\delta u)_{\bar{y}} - g u .$$

$R(u)$ is a linear difference expression for every boundary point whose exact form we shall not give here.

If we arrange $\sum\sum_{G_h} B(u,v)$ with respect to u, we get

$$h^2 \sum\sum_{G_h} B(u,v) = -h^2 \sum\sum_{G_h'} u\,M(v) - h \sum_{\Gamma_h} u\,S(v). \quad (2)$$

$M(v)$ is called the adjoint difference expression of $L(u)$; we have

$$M(v) = (a v_x)_{\bar{x}} + (b v_y)_{\bar{x}} + (c v_x)_{\bar{y}} + (d v_y)_{\bar{y}}$$
$$+ (\alpha v)_{\bar{x}} + (\beta v)_{\bar{y}} - \gamma v_x - \delta v_y - g v ,$$

where $S(v)$ is a difference expression for the boundary corresponding to $R(u)$.

The equations (1), (2) and the following equation

$$h^2 \sum\sum_{G_h'} \big(v\,L(u) - u\,M(v)\big) + h \sum_{\Gamma_h} \big(v\,R(u) - u\,S(v)\big) = 0 \quad (3)$$

are called Green's formulas.

The simplest and most important case occurs if the bilinear form is symmetric, i.e., if the equations $b = c$, $\alpha = \gamma$, $\beta = \delta$ exist. In this case the expression $L(u)$ is identical with its adjoint $M(v)$; we therefore call it the self-adjoint case, and it is readily obtainable from the quadratic expression

$$B(u,u) = a u_x^2 + 2b u_x u_y + d u_y^2 + 2\alpha u_x u + 2\beta u_y u + g u^2$$

In the following we shall limit ourselves mainly to expressions $L(u)$ which are self-adjoint. The character of the difference expression $L(u)$ depends above all on the nature of those terms of the quadratic form $B(u,u)$ which are quadratic in the first difference quotients. We call this part of $B(u,u)$ the characteristic form:

$$P(u,u) = a u_x^2 + 2b u_x u_y + d u_y^2 .$$

Accordingly as $P(u,u)$ is (positive) definite or indefinite in the difference quotients we call the corresponding difference expression $L(u)$ elliptic or hyperbolic.

The difference expression

$$\Delta u = u_{x\bar{x}} + u_{y\bar{y}}$$

with which we shall primarily concern ourselves in the following paragraphs, is elliptic. In particular, it is obtained from the quadratic expression

$$B(u,u) = u_x^2 + u_y^2 \quad \text{or} \quad u_{\bar{x}}^2 + u_{\bar{y}}^2 .$$

Hence the corresponding Green's formulas are[3]

$$h^2 \sum_{G_h}\sum (u_x^2 + u_y^2) = -h^2 \sum_{G_h'}\sum u\Delta u - h \sum_{\Gamma_h} u R(u), \tag{4}$$

$$h^2 \sum_{G_h'}\sum (v\Delta u - u\Delta v) + h \sum_{\Gamma_h} \left(vR(u) - uR(v)\right) = 0. \tag{5}$$

The difference expression $\Delta u = u_{x\bar{x}} + u_{y\bar{y}}$ is obviously the analogue of the differential expression $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ for a function $u(x, y)$ of the continuous variables x and y. Written out completely the difference expression reads

$$\Delta u = \frac{1}{h^2}\left\{u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)\right\}.$$

Therefore $h^2\Delta u/4$ is the excess of the arithmetic mean of the function values at the four neighboring points over the function value at the point in question.

Completely similar considerations lead to linear difference expressions of the fourth order and corresponding Green's formulas if we begin with the bilinear difference expressions which are formed from the difference quotients of second order. We content ourselves with the example of the difference expression

$$\Delta\Delta u = u_{xx\bar{x}\bar{x}} + 2u_{x\bar{x}y\bar{y}} + u_{yy\bar{y}\bar{y}} .$$

3 The boundary expression $R(u)$ may be written here as follows: If $u_0, u_1, \ldots, u_\nu$ be the function values at the boundary point concerned and at its ν neighboring points ($\nu \leq 3$), then

$$R(u) = \frac{1}{h}(u_1 + \cdots + u_\nu - \nu u_0).$$

This arises from the quadratic expression

$$B(u,u) = (u_{x\bar{x}} + u_{y\bar{y}})^2 = (\Delta u)^2$$

if one orders the sum

$$h^2 \sum\sum_{G_h'} \Delta u \Delta v$$

with respect to v, or correspondingly replaces u by the expression Δu in equation (5). One must notice however that in the expression $\Delta\Delta u$, the function value at a point is connected with the function values at its neighboring points and at their neighboring points, and accordingly is defined only for such points of the region G_h which are also interior points of the region G_h', (see (5)); the totality of such points we designate by G_h''. We then obtain Green's formula

$$h^2 \sum\sum_{G_h'} \Delta u \cdot \Delta v = h^2 \sum\sum_{G_h''} v \cdot \Delta\Delta u + h \sum_{\Gamma_h + \Gamma_h'} v \cdot R(u), \qquad (6)$$

where $R(u)$ is a definable linear difference expression for each point of the boundary strip $\Gamma_h + \Gamma_h'$, though we omit its more precise form. Γ_h' indicates in this case the set of boundary points of G_h'.

§ 2. Boundary Value Problems and Eigenvalue Problems

1. The Theory of the Boundary Value Problem. The boundary value problem for linear elliptic homogeneous difference equations of second order, which corresponds to the classical boundary value problem for partial differential equations, can be formulated as follows:

In a grid region G_h let there be given a self-adjoint elliptic linear difference expression of the second order $L(u)$. It may originate from a quadratic expression $B(u, u)$ which is positive definite in the sense that it cannot vanish if u_x and u_y themselves do not vanish.

We now determine in G_h a function u satisfying the difference equation

$$L(u) = 0 ,$$

and which coincides with prescribed values at the boundary points of this grid region.

Our requirement is represented by just as many linear equations and therefore function values that are to be determined, as there are interior grid points of the grid region.[4] Some of these equations are homogeneous, namely, those which correspond to grid points which, with their four neighbors, lie in the interior; others, in which boundary points of the grid region enter, are nonhomogeneous. If we set the right-hand side of the nonhomogeneous system of equations (i.e., the boundary values of u) equal to zero, then it follows at once from Green's formula (1) that $B(u, u)$ vanishes when we set $u = v$; and because of the positive definite character of $B(u, u)$, there also follows the vanishing of u_x, u_y and therefore also of u. Thus the difference equation has the solution $u = 0$ if the boundary values vanish; or, in other words, the solution is uniquely determined, if

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[4] If one considers an arbitrary difference equation of the second order $L(u) = 0$ as a system of linear equations, and constructs the transposed system of equations, then this transposed set is represented by the adjoint difference equation $M(v) = 0$. Thus the above self-adjoint difference equation gives rise to a linear equation system with a symmetric matrix.

at all, by the boundary values, since the difference of two solutions with the same boundary values must vanish. If, however, a linear system of equations with just as many unknowns as equations possesses the property that for vanishing right-hand sides the unknowns themselves must also vanish, then the fundamental theorem of the theory of equations asserts that for an arbitrarily prescribed right-hand side exactly one solution must exist. In our case there follows at once the existence of a solution of the boundary value problem.

We see that for our elliptic difference equations the unique determination and the existence of a solution of the boundary value problem are related to one another by the fundamental theorem of the theory of linear equations, while in the theory of partial differential equations both facts must be proved by quite different methods. The basis for this difficulty is to be found in the fact that the differential equations are no longer equivalent to a finite number of equations, and so one can no longer depend upon the equality of the number of unknowns and equations.

Since the difference equation

$$\Delta u = 0$$

originates from the positive definite quadratic expression

$$h^2 \sum_{G_h}\sum (u_x^2 + u_y^2) ,$$

then the boundary value problem of this difference equation is always uniquely solvable.

The theory for difference equations of higher order, e.g., of fourth order, is developed entirely parallel to that for the difference equations of second order, for which the example of the difference equation

$$\Delta\Delta u = 0$$

may be sufficient. In this case the values of the function u in the boundary strip $\Gamma_h + \Gamma_h'$ must be given. Evidently here also the difference equation $\Delta\Delta u = 0$ yields just as many linear equations as there are unknown function values at the points of G_h''. In order to demonstrate the uniqueness of the solution of the boundary value problem we need only show that a solution which has the value zero in the boundary strip $\Gamma_h + \Gamma_h'$ necessarily vanishes identically. For this purpose we note that the sum over the corresponding quadratic expression

$$h^2 \sum_{G_h''}\sum (\Delta u)^2 \tag{7}$$

for such a function vanishes, as one may notice by transforming the sum according to Green's formula (6). The vanishing of the sum (7), however, causes the vanishing of Δu in all points G_h', and according to the above proof this can only happen for vanishing boundary values if the function u assumes the value zero throughout the region. Thus, our assertion is proven and the uniqueness of the solution of the boundary value problem of the difference expression is guaranteed.[5]

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[5] For a discussion of the solution of our boundary value problem by iterative methods, see, among others, the treatment in: "Über Randwertaufgaben bei partiellen Differenzengleichungen", by R. Courant, Zeitschr. f. angew. Mathematik u. Mechanik, 6 (1925), pp. 322-325. In addition one is referred to the report of H. Henky, Zeitschr. f. angew. Math. u. Mech., 2 (1922), p. 58 ff.

2. Relations to Minimum Problems. The above boundary value problem is related to the following minimum problem: among all functions $\varphi(x, y)$ defined in the grid region G_h which assume prescribed values at the boundary points, there is to be found such a function $\varphi = u(x, y)$ for which the sum over the grid region

$$h^2 \sum_{G_h}\sum B(\varphi, \varphi)$$

assumes the smallest possible value. In this case we assume that the quadratic difference expression of the first order $B(u, u)$ is positive definite in the above sense. (See p. 6.) One may notice that the difference equation $L(\varphi) = 0$ results from this minimum requirement as a restriction on the solution $\varphi = u(x, y)$, where $L(\varphi)$ is the difference expression of the second order derived in the above manner from $B(\varphi, \varphi)$, either by applying the rules of the differential calculus to the sum $h^2 \sum_{G_h}\sum B(\varphi, \varphi)$ as a function of a finite number of values of φ at the grid points, or similarly, by employing the usual methods from the calculus of variations.

By way of example, the boundary value problem of finding a solution of the equation $\Delta\varphi = 0$ which assumes prescribed boundary values is equivalent to the problem of minimizing the sum $h^2 \sum_{G_h}\sum (\varphi_x^2 + \varphi_y^2)$ over all those functions which assume these boundary values.

The situation is entirely similar for difference equations of the fourth order, where again we restrict ourselves to the example of $\Delta\Delta\varphi = 0$. The boundary value problem corresponding to this difference equation is equivalent to the problem of minimizing the sum $h^2 \sum_{G_h'}\sum (\Delta\varphi)^2$

over all those functions $\varphi(x, y)$ which take on the prescribed values in the boundary strip Γ_h'. Besides this sum, still other expressions which are quadratic in the second derivatives yield the equation $\Delta\Delta u = 0$ under the requirement that they are to be minimized, as for example the sum

$$h^2 \sum_{G_h'}\sum \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right),$$

in which there appear second difference quotients exclusively at points of G_h.

That the posed minimum problem always possesses a solution follows from the theorem that a continuous function of a finite number of variables (the functional values of φ at the grid points) must always have a minimum if this function is bounded from below, and if it tends to infinity as soon as at least one of the independent variables tends to infinity.[6]

3. Green's Function. One can treat the boundary value problem of the nonhomogeneous equation $L(u) = -f$ similarly to the boundary value problem of the homogeneous equation $L(u) = 0$. It is sufficient for the case of the nonhomogeneous equation to limit oneself to the case when the boundary values of u vanish everywhere, since we obtain the solution for other boundary values by the addition of a suitable solution of the homogeneous equation. To solve the system of linear equations which is represented by the boundary value problem of

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6 It can easily be verified that the assumptions for the application of this theorem are valid.

$L(u) = -f$, we first choose the function $f(x, y)$ so that at a grid point with the coordinates $x = \xi$, $y = \eta$ it assumes the value $-h^{-2}$, and at all other grid points assumes the value zero. If $K(x, y; \xi, \eta)$ is a solution of the special resulting difference equation which will now depend on the parametric point (ξ, η) and which vanishes at the boundary, then the solution corresponding to an arbitrary function is represented by the sum

$$u(x, y) = h^2 \sum_{\xi, \eta \text{ in } G_h}\sum K(x, y; \xi, \eta) f(\xi, \eta).$$

We call the function $K(x, y; \xi, \eta)$, which depends upon the points (x, y) and (ξ, η), the Green's function of the difference expression $L(u)$. If we denote by $\bar{K}(x, y; \xi, \eta)$ the Green's function of the adjoint expression $M(v)$, then we have the relation

$$K(\bar{\xi}, \bar{\eta}; \xi, \eta) = \bar{K}(\xi, \eta; \bar{\xi}, \bar{\eta}),$$

which can be obtained immediately from Green's formula (5) if one sets $u = K(x, y; \xi, \eta)$ and $v = \bar{K}(x, y; \bar{\xi}, \bar{\eta})$. For a self-adjoint difference expression there results from the above relation the symmetry relation

$$K(\bar{\xi}, \bar{\eta}; \xi, \eta) = K(\xi, \eta; \bar{\xi}, \bar{\eta}).$$

4. Eigenvalue Problems. Self-adjoint difference expressions $L(u)$ give rise to eigenvalue problems of the following type: Let there be sought the values of a parameter λ -- the eigenvalues -- for which the difference equation in G_h

$$L(u) + \lambda u = 0$$

possesses a solution -- the eigenfunction -- which vanishes on the boundary Γ_h.

The eigenvalue problem is equivalent to the principal axis problem of the quadratic form $B(u,u)$. There are just as many eigenvalues $\lambda^{(1)}, \dots, \lambda^{(N)}$ as interior grid points in the region G_h, and just as many corresponding eigenfunctions $u^{(1)}, \cdots, u^{(N)}$. The system of eigenfunctions and eigenvalues and their existence results from the minimum problem:

Among all the functions $\varphi(x,y)$ which vanish on the boundary and satisfy the $(m-1)$ orthogonality conditions

$$h^2 \sum_{G_h}\sum \varphi u^{(\nu)} = 0 \qquad (\nu = 1, \cdots, m-1)$$

and the normalization condition

$$h^2 \sum_{G_h}\sum \varphi^2 = 1 ,$$

we seek that one $\varphi = u$ for which the sum

$$h^2 \sum_{G_h}\sum B(\varphi,\varphi)$$

assumes the smallest value. The value of the minimum is the m^{th} eigenvalue and the function for which it is assumed is the m^{th} eigenfunction.[7]

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[7] Because of the orthogonality of the eigenfunctions, i.e., $h^2 \sum_{G_h}\sum u^{(\nu)} u^{(\mu)} = 0 \ (\nu \neq \mu)$, every grid function $g(x,y)$ which vanishes on the boundary can be developed in a series of eigenfunctions in the form

$$g = \sum_{\nu=1}^{N} c^{(\nu)} u^{(\nu)} ,$$

where the coefficients $c^{(\nu)}$ are obtained from the equation

$$c^{(\nu)} = \sum_{G_h}\sum g u^{(\nu)} .$$

In this way we obtain in particular the following representation of the Green's function:

$$K(x,y;\xi,\eta) = -\frac{1}{h^2} \sum_{\nu=1}^{N} \frac{u^{(\nu)}(x,y)\, u^{(\nu)}(\xi,\eta)}{\lambda^{(\nu)}} .$$

§ 3. Relation to the Problem of a Random Walk[8]

Our topic is related to the question of probability calculation, namely the problem of a random walk in a bounded region.[9] In a grid region G_h one may imagine the grid lines as paths along which a particle can move from a point to a neighboring point. In this path - network our particle may now move at random, choosing by chance at each path junction among the four available directions -- all four are equally probable. The random walk stops as soon as a boundary point of G_h is reached, where the particle may be absorbed.

We shall ask:

1. What is the probability $w(P;R)$ that starting from a point P one can arrive in some time or other by a random path at a boundary point R ?

2. What is the mathematical expectation $v(P;Q)$ that by such a random path starting from P one reaches a point Q of G_h without first meeting the boundary?

We may consider this probability or the mathematical expectation more precisely by the following process. We imagine a unit quantity of some substance existing at a point P. The substance may spread out in our path net with a constant velocity, travelling by chance one grid space in one time unit. At each grid point exactly a quarter of

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8 Here § 3 is unnecessary for the discussion of the limiting process in § 4.

9 There is an essential difference in the way in which the boundaries of the region are introduced in the following consideration and in well-known methods, such as, for example, those which have been used in connection with Brownian molecular motion.

the incoming substance should flow toward each of the four directions. The amount of substance which arrives at a boundary point will be held there. If the source point P is a boundary point, then the entire quantity of substance will remain there.

We interpret in general the probability $w(P;R)$ of reaching the boundary point R by a random walk starting from P without having before touched the boundary, as the quantity of substance which collects after an infinite time at this boundary point.

The probability $E_n(P;Q)$ of reaching a point Q in exactly n steps from point P without touching the boundary, we interpret as the quantity of substance at point Q after n time units, when P and Q are both interior points; if P or Q is a boundary point, then we set it equal to zero.

The value of $E_n(P;Q)$ is just the amount moving in n steps from P toward Q without meeting the boundary, divided by 4; therefore it follows that $E_n(P;Q) = E_n(Q;P)$.

By the mathematical expectation $v(P;Q)$ for the above mentioned random walk going once from P to point Q, we mean in general the infinite sum of all these probabilities,[10]

$$v(P;Q) = \sum_{\nu=0}^{\infty} E_\nu(P;Q) ,$$

which is therefore for interior points P and Q the sum of all quantities of substance which have passed the point Q at different moments

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[10] We shall prove their convergence shortly.

of time. Therefore the expectation value 1 will be assigned for reaching point Q. For boundary values this expectation is equal to zero.

Denoting the quantity arriving at a boundary point R in exactly n steps by $F_n(P;R)$, the probability $w(P,R)$ is thus represented by the infinite sum

$$w(P;R) = \sum_{\nu=0}^{\infty} F_\nu(P;R) ,$$

all of whose members are positive, and whose partial sums can never be greater than one, because the substance arriving at the boundary will make up only a part of the original quantity of substance. But with this the convergence of the series is guaranteed.

One can easily see that the probability $E_n(P;Q)$, i.e., the quantity of substance arriving at a point Q after exactly n steps, tends toward zero with increasing n. For if Q is a point from which a boundary point R will be reached in exactly m steps, and if we have $E_n(P;Q) > \alpha > 0$, then after m steps at least the amount $\alpha 4^{-m}$ arrives at this boundary point R; but, because of the convergence of the sum $\sum_{\nu=0}^{\infty} F_\nu(P;R)$, the quantity of substance arriving at the boundary point R tends toward zero with time, and the value of $E_n(P;Q)$ itself must therefore also tend toward zero with increasing n; i.e., the probability of an infinitely long path remaining in the interior is zero.

From this it follows that the whole quantity of substance must finally arrive at the boundary; in other words, the sum extended over all boundary points R is

$$\sum_R w(P;R) = 1 .$$

We still have to prove the convergence of the infinite series for mathematical expectation $v(P;Q)$,

$$v(P;Q) = \sum_{\nu=0}^{\infty} E_\nu(P;Q).$$

For this purpose we notice that the quantities $E_n(P;Q)$ satisfy the following relation

$$E_{n+1}(P;Q) = \frac{1}{4}\{E_n(P;Q_1) + E_n(P;Q_2)$$
$$+ E_n(P;Q_3) + E_n(P;Q_4)\}, \quad (n \geq 1),$$

where Q_1 to Q_4 are the four neighboring points of the interior point Q. That is, the quantity of substance arriving after $n+1$ steps at point Q is one fourth of the amount of substance reaching the four neighboring points of Q after n steps. If one of the neighboring points of Q is a boundary point, e.g., $Q_1 = R$, then there follows that no quantity of substance flows from this boundary point to Q, since in the expression we have set $E_n(P;R)$ equal to zero. Furthermore for an interior point $E_0(P;P) = 1$, and of course $E_0(P;Q) = 0$.

From these relations we obtain for the partial sums

$$v_n(P;Q) = \sum_{\nu=0}^{n} E_\nu(P;Q)$$

the equations

$$v_{n+1}(P;Q) = \frac{1}{4}\{v_n(P;Q_1) + v_n(P;Q_2) + v_n(P;Q_3) + v_n(P;Q_4)\},$$

if P does not coincide with Q ; otherwise we have

$$v_{n+1}(P;P) = 1 + \frac{1}{4}\left\{v_n(P;P_1) + v_n(P;P_2) + v_n(P;P_3) + v_n(P;P_4)\right\},$$

i.e., the expectation of a particle coming back to its starting point is composed of the expectation of reaching the point P again on a non-disappearing path, namely $\frac{1}{4}\left\{v_n(P;P_1) + v_n(P;P_2) + v_n(P;P_3) + v_n(P;P_4)\right\}$, and from the expectation unity, which expresses the fact that the whole substance was originally present at this point.

Therefore the quantities $v_n(P;Q)$ satisfy the following difference equations:[11]

$$\Delta v_n(P;Q) = \frac{4}{h^2} E_n(P;Q), \quad \text{if } P \neq Q,$$

$$\Delta v_n(P;Q) = \frac{4}{h^2}\left(E_n(P;Q) - 1\right), \text{ if } P = Q.$$

Here $v_n(P;Q)$ is equal to zero if Q is a boundary point.

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11 In this case the Δ-operation is related to the variable point Q. This equation may be interpreted as a type of heat conduction equation. Namely, if one considers the function $v_n(P;Q)$ as a function of time t which is proportional to n so that $t = n\tau$ and $v_n(P;Q) = v(P;Q;t) = v(t)$, instead of as a function of the index n as in the above representation, we can then write the above equations in the following form:

$$\Delta v(t) = \frac{4\tau}{h^2} \frac{v(t+\tau) - v(t)}{\tau} \quad \text{for } P \neq Q,$$

$$\Delta v(t) = \frac{4\tau}{h^2}\left(\frac{v(t+\tau) - v(t)}{\tau} - 1\right) \text{ for } P = Q.$$

For the limiting process of a similar difference equation to a parabolic differential equation see Part II, § 6, p. 55.

The solution of this boundary value problem is, as already demonstrated earlier, uniquely defined for any right-hand side (see p. 10) and depends continuously on the right-hand side. Now since the values of $E_h(P;Q)$ tend toward zero, therefore the solutions $v_h(P;Q)$ converge toward the solutions $v(P;Q)$ of the difference equations

$$\Delta v(P;Q) = 0 \quad , \qquad \text{if} \quad P \neq Q \ ,$$

$$\Delta v(P;Q) = -\frac{4}{h^2} \quad , \qquad \text{if} \quad P = Q \ ,$$

with the boundary values $v(P;R) = 0$.

Therefore we see that the mathematical expectation $v(P;Q)$ exists and is nothing more than the Green's function $K(P;Q)$ belonging to the difference equation $\Delta u = 0$, but supplied with the factor 4. The symmetry of the Green's function $K(P;Q) = K(Q;P)$ is an immediate consequence of the symmetry of the values $E_h(P;Q)$, with whose help it was defined.

The probability $w(P;R)$ satisfies the relation

$$w(P;R) = \frac{1}{4}\left\{w(P_1;R) + w(P_2;R) + w(P_3;R) + w(P_4;R)\right\}$$

with respect to P, and therefore also the difference equation

$$\Delta w = 0$$

Since P_1, P_2, P_3, P_4 are the four neighboring points of the interior point P, therefore each path from P to R must pass through one of these four points, and each of the four path directions is equally probable. Further, the probability of going from one boundary point R to another R' is zero, $w(R, R') = 0$, and if the two points R and R' coincide, we have $w(R, R) = 1$. Therefore $w(P;R)$ is the solution of the boundary value problem $\Delta w = 0$, where the boundary value 1 is prescribed at the boundary point R and the value 0 is prescribed at all other points of the boundary. The solution of the boundary value problem for arbitrary boundary values $u(R)$ then has simply the form $u(P) = \sum_R w(P;R)\, u(R)$, which is to be summed over all boundary points R.[12] If we here substitute for u the function $u \equiv 1$, we thus again obtain the relation $1 = \sum_R w(P;R)$.

The present interpretation of Green's function as expectation permits further properties to be immediately recognized. We mention only the fact that the Green's function increases if one changes from the region G to one which contains G as a partial region; for then the number n of the possible grid paths leading from one point P to another Q without touching the boundary also increases.

Naturally for more than two independent variables corresponding relations hold. We satisfy ourselves with the observation that other elliptic difference equations permit a similar probability interpretation.

If one goes through the limiting process to vanishing mesh width, which is easy to perform by the methods of the following paragraph, then Green's function for the grid changes to within a numerical factor

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12 One can easily see that the probability $w(P;R)$ of reaching the boundary is the boundary expression $R(K(P;Q))$ written with respect to Q with the Green's function $K(P;Q)$, by identifying $u(x, y)$ in Green's formula (5) with $w(P, Q)$ and $v(x, y)$ with $v(P, Q)$.

to the Green's function of the potential equation; a similar relation exists between the expression $h^{-1}w(P;R)$ and the normal derivative of Green's function at the boundary of the region. In this way the Green's function of the potential equation, for example, could be interpreted as the specific mathematical expectation of going from one point to another without touching the boundary.[13]

After the limiting process from the grid to the continuum, the influence of the prescribed grid directions of the random walk vanishes. This fact is of importance when one undertakes the limiting process with a more general random walk problem without prescribed directions, and is in principle an interesting problem; however it exceeds the scope of this discussion, but is one to which we hope to return on another occasion.

§ 4. Limiting Process to the Solution of the Differential Equation

1. The Boundary Value Problem of Potential Theory. In carrying out the limiting process for the solution of the difference equation problems to the solution of the corresponding differential equations, we will not seek the greatest possible generality in the formulation with respect to the boundary and the boundary values themselves, in order to make the characteristics of our method clearer.[14] Accordingly

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[13] Thereby the area of a surface is assigned as the expectation of reaching the surface element.

[14] It might be remarked that the extension of our method to more general boundaries and prescribed boundary values in no way gives rise to fundamental difficulties.

we assume that a simply connected region G is given in the plane, whose boundary consists of a finite number of arcs having continuous tangents. In a region containing G in its interior let there be given a continuous function $f(x, y)$ having continuous partial derivatives of first and second order. For the grid domain G_h, corresponding to the mesh width h and to the region G, let the boundary value problem of the difference equation $\Delta u = 0$ be solved with the same boundary values which are assumed by the function $f(x, y)$ at the boundary points of G_h; call this solution $u_h(x, y)$. We wish to prove that with vanishing mesh width h the grid function u_h converges toward the solution u of the boundary value problem of the partial differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ for the region G, where the boundary values for the region G are again provided by those values which the function $f(x, y)$ assumes on the boundary of G. Furthermore we shall show that for every region lying entirely in the interior of G the difference quotients of u_h of arbitrary order converge uniformly toward the corresponding partial differential quotients of the limiting function $u(x, y)$.

In carrying through the proof of convergence it is convenient to replace the requirement that $u(x, y)$ assumes the boundary values by the following weaker condition: If S_r is that boundary strip of the region G whose points are at a distance less than r from the boundary, then the integral

$$\frac{1}{r}\iint_{S_r}(u-f)^2\,dx\,dy$$

tends toward zero with decreasing r.[15]

Our proof of convergence depends upon the fact that for every subregion G^* lying entirely within the interior of the region G, the function $u_h(x, y)$ and every difference quotient remains bounded with decreasing h and are "equi-continuous" in the following sense: There exists for each of these functions $w_h(x, y)$ a quantity $\delta(\epsilon)$ dependent only on the region and not on h such that

$$|w_h(P) - w_h(P_1)| < \epsilon,$$

when the grid point P and P_1 of the grid region G_h lie in the given subregion, and are at a distance less than $\delta(\epsilon)$ from one another.

Once we have proved the asserted equi-continuity then we can select by known methods a subsequence of our functions u_h in such a way that, together with its difference quotients of every order, it converges uniformly in every subregion G^* toward the limiting function

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15 Note that our weaker boundary value requirement actually provides the unique characterization of the solution, as follows from a theorem which is easy to prove: If, for a function satisfying the differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ in the interior of G, the above form of the boundary condition is satisfied with $f(x, y) = 0$, and if $\iint((\partial u/\partial x)^2 + (\partial u/\partial y)^2)\,dx\,dy$ exists, then $u(x, y)$ is identically zero. (See Courant, "Über die Lösungen der Diff. Gl. der Physik", Math. Annalen, 85, especially pp. 296 ff.)

In the case of two independent variables the fact that these boundary values are actually taken on can be concluded from our weaker requirement; in the case of more variables one cannot expect the corresponding result in general, because on the boundary there may be exceptional points at which boundary values need not be assumed, while for the weaker requirement a solution still exists.

$u(x, y)$ or its differential quotients, respectively. The limiting function possesses corresponding derivatives of arbitrarily high order in each interior subregion G^* of G, and there satisfies the partial differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. If we then show that it satisfies the boundary conditions, we recognize it in the solution of our boundary value problem for the region G. Since this solution is uniquely determined then it is further shown that not only a subsequence of the functions u_h but that this sequence of functions itself also possesses the asserted convergence property.

The equi-continuity of our quantities follows from the demonstration of the following results:

1. As h decreases the sums

$$h^2 \sum_{G_h}\sum u^2 \quad , \quad h^2 \sum_{G_h}\sum (u_x^2 + u_y^2)$$

extended over the grid region G_h are bounded.[16]

2. If $w = w_h$ satisfies the difference equation $\Delta w = 0$ in a grid region G_h, and if with decreasing h the sum

$$h^2 \sum_{G_h^*}\sum w^2 \quad ,$$

extended over a grid region G_h^* corresponding to a subregion G^* of G is bounded, then for every fixed subregion G^{**} lying entirely in the interior of G^* the sum

$$h^2 \sum_{G_h^{**}}\sum (w_x^2 + w_y^2) \quad ,$$

16 Here and when later convenient we shall drop the subscript h from grid functions.

extended over the corresponding grid region G_h^{**} remains bounded with decreasing h.

Together with 1. it follows from this that, since all of the difference quotients w of the function u_h satisfy the difference equation $\Delta w = 0$, each of the sums

$$h^2 \sum_{G_h^*}\sum w^2$$

is bounded.

3. From the boundedness of these sums there follows finally the boundedness and equi-continuity of all the difference quotients themselves.

2. Proof of the Lemmas. The proof of theorem 1. follows from the fact that the function values u_h are themselves bounded. Since the largest and smallest value of the function will be assumed on the boundary[17], it therefore tends toward prescribed finite values. The boundedness of the sum $h^2 \sum_{G_h}\sum (u_x^2 + u_y^2)$ is an immediate result of the minimum property of our grid function formulated in § 2, 2, whereby

$$h^2 \sum_{G_h}\sum (u_x^2 + u_y^2) \leq h^2 \sum_{G_h}\sum (f_x^2 + f_y^2)$$

certainly holds. The sum on the right converges with decreasing mesh

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17 We remark explicitly that with regard to the application of the method to other differential equations, we need not use this condition. In this connection we could use the inequality (15) or employ an alternative method. (Translator's note: see Courant and Hilbert, Methoden der mathematischen Physik, Vol. 1, Chapt. III, 3, "Alternative to the Theory of Integral Equations").

width toward the integral $\iint\limits_G \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right] dx\,dy$, which we have assumed to exist.

In order to prove the lemmas formulated under 2, we consider the quadratic sum

$$h^2 \sum_{Q_1}\sum \left(w_x^2 + w_{\bar{x}}^2 + w_y^2 + w_{\bar{y}}^2 \right),$$

where the sum is to extend over all interior points of a square Q_1 (see Fig. 1). We designate the function values on the outer boundary of the square Q_1 by w_1, those on the second boundary line S_0 by w_0. Then

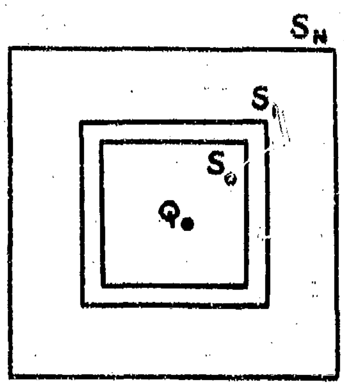

Fig. 1.

Green's formula yields

$$h^2 \sum_{Q_1}\sum \left(w_x^2 + w_{\bar{x}}^2 + w_y^2 + w_{\bar{y}}^2 \right) = \sum_{S} \left(w_1^2 - w_0^2 \right) \leq \sum_{S_1} w^2 - \sum_{S_0} w^2, \quad (8)$$

where the summation on the right is to extend over the two outer boundary lines S_1 and S_0, and where w_1 and w_0 refer to neighboring points. We now consider a set of concentric squares Q_0, Q_1, Q_2, ..., Q_N with boundaries S_0, S_1, ..., S_N each of which arises from the previous one in such a way that the border of nearest neighboring points is added. (See Fig. 1.) To each of these squares we apply the appraisal (8) and notice that

$$2h^2 \sum_{Q_0}\sum (w_x^2+w_y^2) \leq h^2 \sum_{Q_k}\sum (w_x^2+w_{\bar{x}}^2+w_y^2+w_{\bar{y}}^2)$$

holds for $k \geq 1$. If we add the series of n inequalities

$$2h^2 \sum_{Q_0}\sum (w_x^2+w_y^2) \leq \sum_{S_{k+1}} w^2 - \sum_{S_k} w^2 \qquad (0 \leq k < n),$$

we obtain

$$2nh^2 \sum_{Q_0}\sum (w_x^2+w_y^2) \leq \sum_{S_n} w^2 - \sum_{S_0} w^2 \leq \sum_{S_n} w^2 .$$

This inequality we sum from $n = 1$ to $n = N$. One thus obtains

$$N^2 h^2 \sum_{Q_0}\sum (w_x^2+w_y^2) \leq \sum\sum w^2 ,$$

in which the sum on the right extends over the entire square Q_N.

By reduction of the mesh width we now let the squares Q_0 and Q_N tend toward two fixed concentric squares lying in the interior of G and a distance a apart, so that Nh converges toward a and we find that

$$h^2 \sum_{Q_0}\sum (w_x^2+w_y^2) \leq \frac{h^2}{a^2} \sum_{Q_N}\sum w^2 \tag{9}$$

holds independently of the mesh width.

This inequality -- for sufficiently small mesh width -- holds of course not only for the two squares Q_0 and Q_N, but, with another constant a, for any two subregions of G such that one is contained entirely within the other. Therefore the theorem of 2. is proved.[18]

In order to prove the third result, that in each interior subregion the function u_h and all of its difference quotients w_h remain bounded and equi-continuous with the refinement of the mesh width, we consider a rectangle R with corner points P_0, Q_0, P, Q (see Fig. 2), whose sides $P_0 Q_0$ and PQ are parallel to the x-axis and have the length a.

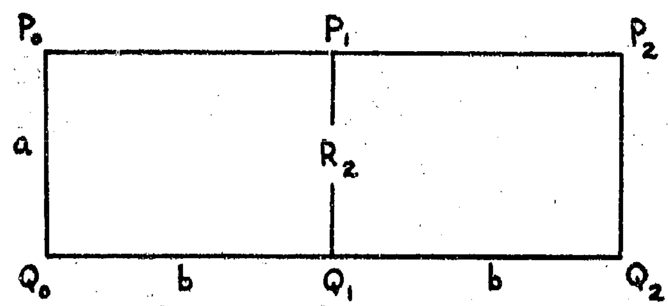

Fig. 2

We start with the relation

$$w(Q_0) - w(P_0) = h \sum_{PQ} w_x + h^2 \sum_R \sum w_{xy}$$

and with the inequality which follows directly

$$\left| w(Q_0) - w(P_0) \right| \leq h \sum_{PQ} |w_x| + h^2 \sum_R \sum |w_{xy}| . \qquad (11)$$

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18 If we do not assume that $\Delta w = 0$, then in place of the inequality (9) we find

$$h^2 \sum_{G^{**}} \sum (w_x^2 + w_y^2) \leq c_1 h^2 \sum_{G^*} \sum w^2 + c_2 h^2 \sum_{G^*} \sum (\Delta w)^2 \qquad (10)$$

for suitable constants c_1 and c_2 independent of h, and where G^{**} lies entirely in the interior of the region G^*, which in turn is contained in the interior of G.

We then let the side PQ of the rectangle vary between an initial line $P_1 Q_1$ a distance b from $P_0 Q_0$ and a final line $P_2 Q_2$ a distance $2b$ from $P_0 Q_0$, and we sum the corresponding $(\frac{b}{h}+1)$ inequalities (11).

Thus we obtain the result

$$\left| w(P_0) - w(Q_0) \right| \leq \frac{1}{b+h} h^2 \sum_{R_2}\sum |w_x| + h^2 \sum_{R_2}\sum |w_{xy}|,$$

in which we extend the summation over the entire rectangle $R_2 = P_0 Q_0 P_2 Q_2$. From Schwarz's inequality there follows

$$\left| w(P_0) - w(Q_0) \right| \leq \frac{1}{b} \sqrt{2ab} \sqrt{h^2 \sum_{R_2}\sum w_x^2} + \sqrt{2ab} \sqrt{h^2 \sum_{R_2}\sum w_{xy}^2}. \quad (12)$$

Since by assumption the sums which occur here multiplied by h^2 remain bounded, it follows that the difference $|w(P_0) - w(Q_0)|$ tends to zero with the distance a, independently of the mesh size since we can fix b for each subregion G^* of G. The equi-continuity of $w = w_y$ in the y-direction is thus proved. Correspondingly one may prove the same for the x-direction, and hence for each interior subregion G^* of G. The boundedness of the function w_h in G^* follows finally from its equi-continuity and from the boundedness of $h^2 \sum_{G^*}\sum w_h^2$.

With this proof one then assures the existence of a subsequence of functions u which converges toward a limiting function $u(x, y)$, and which does so uniformly together with all its difference quotients in the sense discussed above for every interior subregion of G. This limiting function $u(x, y)$ has continuous partial differential quotients of arbitrary order throughout G , and there satisfies the partial differential equation of the potential

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

3. <u>The Boundary Condition</u>. To prove that the solution satisfies the previously stated boundary condition, we next show that for every grid function v we have the inequality

$$h^2 \sum_{S_{r,h}}\sum v^2 \leq A r^2 h^2 \sum_{S_{r,h}}\sum (v_x^2 + v_y^2) + B r h \sum_{\Gamma_h} v^2 , \qquad (13)$$

where $S_{r,h}$ designates that part of the grid region G which lies inside a boundary strip S_r. This boundary strip S_r (see p. 25) consists of all points of G whose distance from the boundary is less than r ; it is confined between Γ and a curve Γ_r . Furthermore A and B are constants dependent only on the region G , and not on the mesh size h nor on the function v .

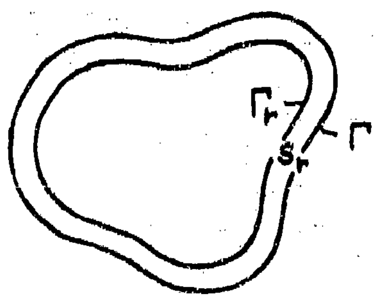

Fig. 3.

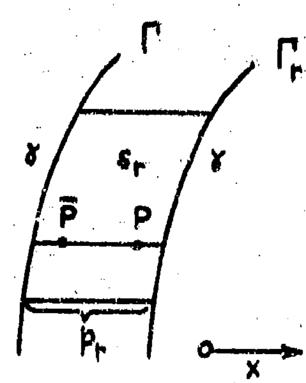

Fig. 4.

In order to prove the above inequality we divide the boundary Γ of G into a finite number of sections, for which the angle between the tangent and either the x-axis or the y-axis stays above some positive value (such as 30°). In illustration let γ be such a section of Γ which is rather steeply inclined to the x-axis. (See Fig. 4.) Lines parallel to the x-axis intersecting the end points of the section γ will cut a section γ_r from the neighboring curve Γ_r, and will define together with γ and γ_r a region s_r of the boundary strip S_r. That part of the grid region G_h which is contained in the strip S_r is called $s_{r,h}$, and the corresponding part of the boundary Γ_h is called γ_h.

We now imagine that a line parallel to the x-axis is drawn through a grid point P_h of $s_{r,h}$. This line will hit the boundary γ_h at a point $\bar{P}_h$. That particular section of this line which lies in $s_{r,h}$ we shall call $p_{r,h}$. Certainly the section will have a length smaller than cr, since r is the largest perpendicular distance of a point on s_r from Γ. Thus the constant c depends only on the smallest angle which the tangent to γ makes with the x-axis.

The following relation holds between the values of v at the point P_h and its value at the point $\bar{P}_h$:

$$v(P_h) = v(\bar{P}_h) \pm h \sum_{P_h \bar{P}_h} v_x ,$$

from which, by squaring and using Schwarz's inequality, one finds

$$v(P_h)^2 \leq 2v(\bar{P}_h)^2 + 2crh \sum_{p_{r,h}} v_x^2 .$$

By summing with respect to P_h in the x-direction we obtain

$$h\sum_{p_r} v^2 \leq 2cr\, v(\bar{P}_h)^2 + 2c^2r^2 h \sum_{p_r} v_x^2 .$$

Summing once more in the y-direction yields the relation

$$h\sum\sum_{S_r} v^2 \leqslant 2cr\sum_{\Gamma_h} v(\bar{P}_h)^2 + 2c^2r^2 h\sum\sum_{S_r} v_x^2 , \tag{14}$$

and in order to find the desired inequality (13) from this one needs only to set up the corresponding expression for the other parts γ of Γ, and then to add the two expressions.[19]

We next set

$$v_h = u_h - f_h$$

so that v_h vanishes on the boundary Γ_h. Then, since $h^2\sum\sum_{G_h}(v_x^2+v_y^2)$ remains bounded for decreasing h, we find from (13)

$$h^2 r^{-1}\sum\sum_{S_{r,h}} v^2 \leq \kappa r , \tag{16}$$

where κ is a constant independent of both the function v and the mesh width. If we extend the summation on the left hand side not over the entire boundary strip $S_{r,h}$ but rather over the difference of two such strips $S_{r,h} - S_{\rho,h}$, then the inequality (16) is still valid with the same constant κ, and we can carry out the limiting process for vanishing mesh width. From the inequality (16), there

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[19] The same sort of considerations which led to the proof of the inequality (13) also yields the inequality

$$h^2\sum\sum_{G_h} v^2 \leq c_1 h\sum_{\Gamma_h} v^2 + c_2 h^2\sum\sum_{G_h}(v_x^2+v_y^2), \tag{15}$$

where the constants c_1 and c_2 depend only on the region G and not on the mesh width.

then results

$$\frac{1}{r}\iint\limits_{S_r - S_\varrho} v^2\,dx\,dy \leq \kappa r\,, \quad v = u - f\,.$$

If we now let the smaller boundary strip S_ϱ approach the boundary, then we find the following inequality:

$$\frac{1}{r}\iint\limits_{S_r} v^2\,dx\,dy = \frac{1}{r}\iint\limits_{S_r} (u-f)^2\,dx\,dy \leq \kappa r\,,$$

which just states that the limiting function u satisfies the boundary conditions which we have specified.

4. Applicability of the Method to other Problems. Our method is essentially based on the inequality (10) stated in the above lemma,[20] because from it the last two theorems of pages 26-7 follow. Inequality (10) in no way makes use of special solutions or other special properties of our difference expressions, and may therefore be immediately extended to the case of arbitrarily many independent variables, or to the eigenvalue problem of the differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \lambda u = 0$, and we thereby obtain exactly the same results concerning convergence properties as above.[21] With some modifications which are easily made the method can be extended to linear differential equations of other sorts, particularly to those with non-constant coefficients. The essential distinction consists only in the proofs of the boundedness

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20 Concerning the application of corresponding integral inequalities, see K. Friedrichs, "Die Rand- und Eigenwertprobleme aus der Theorie der elastischen Platten", Math. Annalen, 98, p. 222.

21 It is thus proved at the same time that each solution of such a differential equation has derivatives of all orders.

of $h^2 \sum\sum u_h^2$; indeed, this sum is not bounded some for linear problems of this type. But when this sum is unbounded, it can in fact be shown that the general boundary value problem for the differential equation concerned has no solution, and that therefore in this case non-vanishing solutions of the associated homogeneous problem, i.e., eigen-functions, exist.[22]

5. The Boundary Value Problem of $\Delta\Delta u = 0$. In order to show that the method can also be applied in the case of differential equations of higher order, we consider in the following the boundary value problem of the differential equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 .$$

We seek a solution of this partial differential equation in our region G, for which the function values and their first derivatives are prescribed on the boundary; these values are themselves defined on the boundary by a previously given function $f(x, y)$. In this case we assume as before (p. 24) that $f(x, y)$ is continuous with first and second derivatives in the plane region which contains the region G.

We replace our differential equation with the problem of solving the difference equation $\Delta\Delta u = 0$ for the grid region G,

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22 See Courant-Hilbert, Methoden der mathematischen Physik, 1, Chap. III, 3, where the theory of integral equations is discussed with the aid of a corresponding alternative principle. See also the Göttingen dissertation of W. v. Koppenfels (to appear).

where the function u assumes the same values as the previously given $f(x, y)$ at the points of the boundary line $\Gamma_h + \Gamma_h'$. From § 2 we know that the boundary value problem for G_h is solvable in one and only one way. With the refinement of the mesh width h we shall show that in each interior sub-region of G this solution converges toward the solution of our differential equation, with all difference quotients approaching their respective differential quotients.

For this purpose we first notice that for the solution $u = u_h$ of our difference problem the sum

$$h^2 \sum_{G_h'}\sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

is bounded with decreasing mesh width. For according to the minimum property of the solution of our difference problem (see p. 12), this sum is never larger than the corresponding sum

$$h^2 \sum_{G_h'}\sum (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2),$$

and this is convergent with the refinement of the mesh width toward the integral

$$\iint_G \left(\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \right) dx\,dy ,$$

which exists according to our assumptions.

From the boundedness of the sum

$$h^2 \sum_{G_h'}\sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

there immediately follows the boundedness of $h^2 \sum_{G_h'}\sum (\Delta u)^2$, as well as of $h^2 \sum_{G_h'}\sum (u_x^2 + u_y^2)$ and $h^2 \sum_{G_h}\sum u^2$. For arbitrary

w there exists the inequality

$$h^2 \sum\sum_{G_h} w^2 \leq c h^2 \sum\sum_{G_h} (w_x^2 + w_y^2) + ch \sum_{\Gamma_h} w^2 \qquad (15)$$

(see (15) p. 34). If one substitutes the first difference quotients of w into this inequality for the function w, and makes use of the subregions of G_h which are defined with these difference quotients, there results the additional inequality

$$h^2 \sum\sum_{G_h} (w_x^2 + w_y^2) \leq c h^2 \sum\sum_{G_h'} (w_{xx}^2 + 2 w_{xy}^2 + w_{yy}^2) + ch \sum_{\Gamma_h + \Gamma_h'} (w_x^2 + w_y^2),$$

where the constant c again does not depend upon the function or the mesh width. We now apply these inequalities with $w = u_h$ and recall in this case the boundedness of the sums over $\Gamma_h + \Gamma_h'$ on the right hand side; the boundary sums converge by definition toward the corresponding integrals formed with $f(x, y)$. Therefore from the boundedness of

$$h^2 \sum\sum_{G_h'} (u_{xx}^2 + 2 u_{xy}^2 + u_{yy}^2)$$

there follows the boundedness of

$$h^2 \sum\sum_{G_h} (u_x^2 + u_y^2) \quad \text{and} \quad h^2 \sum\sum_{G_h} u^2 .$$

We now substitute successively for w the expressions Δu, Δu_x, Δu_y, Δu_{xx}, ... in the inequality

$$h^2 \sum\sum_{G^{**}} (w_x^2 + w_y^2) \leq c h^2 \sum\sum_{G^*} w^2 + c h^2 \sum\sum_{G^*} (\Delta w)^2 \qquad (10)$$

(see p. 30), where G^* is a subregion of G containing G^{**} in its interior, all of which satisfy the equation $\Delta w = 0$. It then follows that for all interior subregions G^* of G, the sums

$$h^2 \sum\sum_{G^*} (w_x^2 + w_y^2) ,$$

i.e.,

$$h^2 \sum_{G^*}\sum (\Delta u_x^2 + \Delta u_y^2), \quad h^2 \sum_{G^*}\sum (\Delta u_{xx}^2 + \Delta u_{xy}^2), \ldots,$$

are bounded, together with the sums

$$h^2 \sum_{G_h}\sum u^2, \qquad h^2 \sum_{G_h}\sum (u_x^2 + u_y^2),$$

and

$$h^2 \sum_{G_h}\sum (\Delta u)^2,$$

which have previously been shown to be bounded.

Finally we substitute successively for w the functions u_{xx}, u_{xy}, u_{yy}, u_{xxx}, ... in the inequality (10),

$$h^2 \sum_{G_h^*}\sum (\Delta w)^2, \text{ i.e., } h^2 \sum_{G_h^*}\sum (\Delta u_{xx})^2, \ldots,$$

which remain bounded as just proved. We then recognize that for all subregions the sums

$$h^2 \sum_{G_h^*}\sum (u_{xxx}^2 + u_{xxy}^2), \quad h^2 \sum_{G_h^*}\sum (u_{xyx}^2 + u_{xyy}^2), \ldots,$$

also remain bounded.

From this fact we conclude as on pp. 30 ff. that we may choose a partial sequence from our set of grid functions for which all difference quotients converge uniformly in each interior subregion of G toward a continuous limiting function in the interior of G, which are their respective differential quotients.

We still have to show that this limiting function, which obviously satisfies the differential equation $\Delta\Delta u = 0$, in addition fulfills the prescribed boundary conditions. In this case we satisfy ourselves as above by expressing the boundary conditions in the form[23]

$$\iint_{S_r} (u-f)^2 \,dx\,dy \leq cr^2, \quad \iint_{S_r} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} - \frac{\partial f}{\partial y} \right)^2 \right] dx\,dy \leq cr^2.$$

That the limiting function satisfies these conditions may be shown by applying the same procedure used previously on p. 34 for the function u and its first difference quotients.

On account of the unique determination of the solution in our boundary value problem, we can now recognize in addition that the indicated convergence properties are possessed not only by the selected partial sequence but the function sequence u itself.

II. THE HYPERBOLIC CASE

§ 1. The Equation of the Vibrating String

In this second part of the article we are concerned with initial value problems of hyperbolic linear differential equations, and we shall show that under certain assumptions the solutions of the corresponding difference equations converge toward the solutions of the differential equation during refinement of the mesh width of a basic grid.

We can most simply demonstrate the present relations with the obvious example of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 . \tag{1}$$

In this case we restrict ourselves to the initial value problem in

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23 It is not difficult to show that boundary values for the function and its derivatives are sufficient. See the corresponding observations by K. Friedrichs, loc. cit.

which the values of the function u and its derivatives on the straight line $t=0$ are given.

To obtain the corresponding difference equation we construct a uniform square grid of mesh width h in the x, t -plane. We substitute for the differential equation (1) the difference equation

$$u_{t\bar{t}} - u_{x\bar{x}} = 0 ,$$

with the notations of pp. 3-4. If we choose a grid point P_0 , the difference equation in this case connects the value of the function u at this point with the values at the four neighboring points. If we again denote the four neighboring values by the four indices 1, 2, 3, 4 (see fig. 5), the difference equation takes the simple form

$$u_1 + u_3 - u_2 - u_4 = 0 . \qquad (2)$$

In this case the value of the function u at the point P itself does not appear in the equation.

We imagine the grid to be divided into two different partial grids, as indicated in fig. 5 with circles and crosses. The difference equation then connects with one another only the values of the function

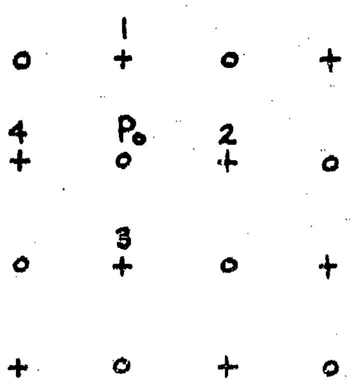

Fig. 5

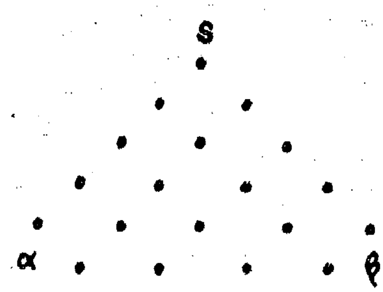

Fig. 6

in each of the partial grids. We therefore restrict ourselves to one of these two partial grids. As initial conditions we have here to prescribe the values of the function u on the two grid rows $t=0$ and $t=h$. Next we state explicitly the solution of this initial value problem; i.e., we express the value of the solution at some point S in terms of the specified values on the two initial rows. One can immediately see that the value at a point on the row $t=2h$ is uniquely defined, merely through the three values connected with it on the first two rows. The value at a point on the fourth row is uniquely defined by the value of the solution at three fixed points of the second and third rows, and therefore by certain values on the first two rows. In general, a certain domain of dependence on the first two rows will belong to a point S; one can find it by drawing the lines $x+t=$ const. and $x-t=$ const. through the point S, until they meet the second row in the points α and β (see fig. 6). We thus call the triangle $S\alpha\beta$ the determination triangle, because within it all u-values do not change as soon as they are specified on the first two rows. We call the lateral lines of the triangle determination lines.

Denoting now the differences of u along the determination lines by $\acute{u}$ and $\grave{u}$, or more precisely

$$\acute{u}_1 = u_1 - u_4 \ , \quad \grave{u}_1 = u_1 - u_2 \ ,$$
$$\acute{u}_2 = u_2 - u_3 \ , \quad \grave{u}_4 = u_4 - u_3 \ ,$$

the difference equation takes the form

$$\acute{u}_1 = \acute{u}_2$$

This means that along a determination line the differences in the other determination direction are constant, and therefore equal to one of the predetermined differences between two points of the first two rows. On the other hand, the difference $u_S - u_\alpha$ is a sum over the differences $\acute{u}$ along the determination line $S\alpha$, so that we obtain, by application of the previous remark, the final formula

$$u_S = u_\alpha + \sum_{\alpha_1}^{\beta_1} \acute{u} \quad , \tag{3}$$

in which the notation is easily understood.

Now we let the mesh width h tend toward zero, whereby the previously mentioned values on the second or first row converge uniformly toward a twice continuously differentiable function $f(x)$, and the difference quotients $\frac{\acute{u}}{h\sqrt{2}}$ tend toward a continuous differentiable function $g(x)$. In this case the right side of (3) obviously transforms uniformly to the expression

$$f(x-t) + \frac{1}{\sqrt{2}} \int_{x-t}^{x+t} g(\xi)\, d\xi \quad , \tag{4}$$

if S converges toward the point (t, x). This is the well know expression of the solution of the wave equation (1) with the initial values $u(x,0) = f(x)$ and $\partial u/\partial t\,(x,0) = f'(x) + \sqrt{2}\, g(x)$. It is therefore shown that the solutions of our difference equation problem converge toward the solution of the differential equation problem with decreasing mesh width, if we let the initial values converge in the above prescribed manner.

§ 2. On the Influence of the Selection of the Grid

The Regions of Dependence for Difference and Differential Equation

The following thoughts are obvious in view of the considerations of § 1. Just as only a certain part of the initial values are decisive for the solution of a linear hyperbolic differential equation at a point S, namely that "region of dependence" cut out by characteristics through S, there is likewise in the solution of a difference equation at the point S a certain dependence region which one obtains by drawing the determination lines from point S. In § 1 the directions of the determination lines of the difference equation coincided with the directions of the characteristics of the differential equation, so that the dependence regions also agree in the limit. This fact, however, depended essentially upon the orientation of the grid in the (x, t)-plane and was based on the fact that we had chosen a square grid. We now take as a basis a general rectangular grid whose mesh width in the t-direction (time-mesh) equals h and in the x-direction (space-mesh) is equal to κh, with constant κ. The region of dependence of the difference equation $u_{t\bar{t}} - u_{x\bar{x}} = 0$ for this grid will be completely in the interior of the region of dependence of the differential equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0$, or will contain the latter in its interior, according as $\kappa < 1$ or $\kappa > 1$.

From this follows a remarkable fact: In the case $\kappa < 1$ as the mesh width h decreases toward zero, the solution of the difference equation cannot converge in general toward the solution of the differential equation. For if in the wave equation (1), one changes the initial values

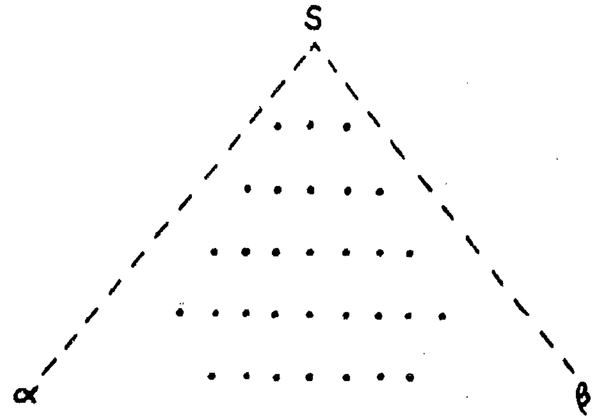

Fig. 7

of the solution of the differential equation in the neighborhood of the end points α and β of the region of dependence (see fig. 7), formula (4) shows that the solution itself also changes at the point (x, t). But for the solutions of the difference equation at the point S the initial values at the points α and β are irrelevant, because they are outside of the region of dependence of the difference equation. -- We shall prove that there is convergence in the case $\varkappa > 1$ in § 3. In this connection see fig. 9, p. 46.

On the other hand, consider as an example the differential equation

$$2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (5)$$

in the spatial coordinates x, y and the time coordinate t, and replace it by the corresponding difference equations in a rectangular grid. In contrast to the case of only two independent variables, it is now impossible to choose the mesh spacing so that the regions of dependence of the difference and differential equations coincide, for the region of dependence of the difference equation is a rectangle, while that of the differential equation is a circle. Later we shall select (see § 4) the mesh spacing so that the region of influence of the difference equation contains the region of influence of the differential equation in its interior, and we shall show that convergence then occurs.

In general the principal result of this section is that one can choose the grid for each linear hyperbolic homogeneous differential equation of second order such that the solution of the difference equation converges toward the solution of the differential equation as the mesh width tends toward zero. (In this connection see §§ 3, 4, 7, 8.)

§ 3. The Passage to the Limit for Arbitrary Rectangular Grids

Next now again consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \ , \tag{1}$$

and select a rectangular grid whose time mesh width is h and whose space mesh is κh. The corresponding difference equation is

$$L(u) = \frac{1}{h^2}(u_1 - 2u_0 + u_3) - \frac{1}{\kappa^2 h^2}(u_2 - 2u_0 + u_4) = 0, \tag{6}$$

where the indices denote the central point P_0 and the corners P_1 , P_2 , P_3 , P_4 of an "elementary rhombus" (see fig. 8). According to the

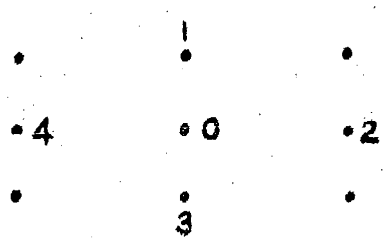

Fig. 8

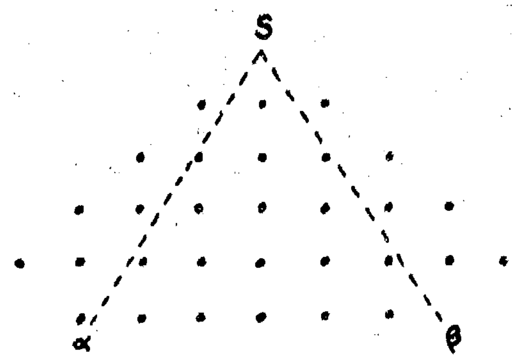

Fig. 9

equation $L(u)=0$ we are able to describe the value of the function u at a point S in terms of its values on that portion of the two rows $t=0$ and $t=h$, (see fig. 6, p. 41) obtained by drawing the two "determination lines" from the point S to the sides of an elementary rhombus. We assume the initial values are prescribed such that they, and the first difference quotients formed from them, converge uniformly with decreasing mesh width and with fixed x toward the continuous prescribed functions on the straight line $t=0$. It is possible to write an explicit solution of the difference equation in terms of the initial values (corresponding to (3) in § 1); but it is not so simple to perform immediately the limiting transition to vanishing mesh width. We therefore follow another course, which will still enable us to treat the general problem.[24]

We multiply the difference expression $L(u)$ by (u_1-u_3) and write the product under consideration according to the following identities:

$$(u_1-u_3)(u_1-2u_0+u_3)=(u_1-u_0)^2-(u_0-u_3)^2, \tag{7}$$

$$(u_1-u_3)(u_2-2u_0+u_4)=(u_1-u_0)^2-(u_0-u_3)^2 \\ -\tfrac{1}{2}\left[(u_1-u_2)^2+(u_1-u_4)^2-(u_2-u_3)^2-(u_4-u_3)^2\right]. \tag{8}$$

[24] In the following compare K. Friedrichs and H. Lewy, "Über die Eindeutigkeit u.s.w.", Math. Annalen, 98, (1928), pp. 192 ff., where the analogous transformations for integrals are used.

We thus obtain

$$2(u_1-u_3)L(u) = \frac{2}{h^2}\left(1-\frac{1}{k^2}\right)\left[(u_1-u_0)^2-(u_0-u_3)^2\right]$$

$$+\frac{1}{k^2h^2}\left[(u_1-u_2)^2+(u_1-u_4)^2-(u_2-u_3)^2-(u_4-u_3)^2\right]. \qquad (9)$$

Now we sum the product (9) over all elementary rhombi of a determination triangle $S\alpha\beta$. On the right side of (9) the squares of the differences in two adjacent elementary rhombi always appear with different signs. They drop out in the summation when both elementary rhombi belong to the triangle $S\alpha\beta$; therefore only one sum remains on the boundary of the triangle. In this way we obtain the relation

$$h^2\sum_{S\alpha\beta}\sum 2\,\frac{u_1-u_3}{h}\,L(u) = h\sum_{S\alpha}\left[2\left(1-\frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2+\frac{1}{k^2}\left(\frac{\acute{u}}{h}\right)^2\right]$$

$$+h\sum_{S\beta}\left[2\left(1-\frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2+\frac{1}{k^2}\left(\frac{\grave{u}}{h}\right)^2\right]$$

$$-h\sum_{I\,II}\left[2\left(1-\frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2+\frac{1}{k^2}\left(\frac{\acute{u}}{h}\right)^2+\frac{1}{k^2}\left(\frac{\grave{u}}{h}\right)^2\right]. \qquad (10)$$

Here $\acute{u}$ and $\grave{u}$ denote differences along determination directions as in §1, while $\dot{u}$ denotes the difference of the function values at two neighboring points connected by a line parallel to the t-axis. The sums are to extend over all boundary lines consisting of two parallel rows, so that all differences $\acute{u}$, $\grave{u}$, $\dot{u}$ occur once and only once.

Therefore the right side of (10) disappears for a solution of $L(u)=0$. The sum which occurs over the initial rows I and II remains bounded if we allow the mesh width h (with fixed κ) to decrease toward zero; in particular it transforms into an integral of the given function on the initial line. Consequently the sums over $S\alpha$ and $S\beta$ in (10) also remain bounded. It is now that we must require $\varkappa \geqq 1$, (see page 44) so that $1 - \frac{1}{\varkappa^2}$ is non-negative and the boundedness of the individual sums

$$h \sum_{S\alpha} \left(\frac{\acute{u}}{h} \right)^2 , \quad h \sum_{S\beta} \left(\frac{\grave{u}}{h} \right)^2$$

follows which we can imagine to extend over arbitrary determination lines.

From this we can derive the "equi-continuity" of a sequence of the grid functions in all directions of the plane (see first part of § 4);[25] because the values of u are bounded on the initial line, there follows the existence of a partial sequence which converges uniformly toward a limiting function $u(x, t)$.

In addition to the function u, its first and second difference quotients also satisfy the difference equation $L(u) = 0$. The initial values of these difference quotients are expressed by means of the equation $L(u) = 0$ by the first, second and third difference quotients of u in which only points on the two initial rows I and II appear. We require that they tend toward continuous limiting functions, i.e., that the given initial values $u(x, 0)$, $u_t(x, 0)$ are continuous and can be differentiated three and two times with respect to x, respectively.

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25 If S_1 and S_2 are two points separated by the distance δ, and if one connects them with a path formed by two lines $S_1 S$ and $S S_2$, the first of which is parallel to one determination direction and the second is parallel to the other, then there results the relation

$$|u_{S_1} - u_{S_2}| \leq |u_{S_1} - u_S| \pm |u_S - u_{S_2}|$$
$$\leq \sqrt{\delta} \sqrt{h \sum_{S_1 S} \left(\frac{\acute{u}}{h}\right)^2} + \sqrt{\delta} \sqrt{h \sum_{S S_2} \left(\frac{\grave{u}}{h}\right)^2} .$$

We can now apply the above convergence properties to the first and second difference quotients of u instead of to u itself, and we can therefore choose a partial sequence such that these difference quotients tend uniformly toward functions which must then be the first or second derivatives of the limiting function $u(x, t)$. The limiting function u consequently satisfies the differential equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$, which corresponds to the difference equation $L(u) = 0$; it thus describes the solution of the initial value problem. Since this solution is uniquely determined, each partial sequence of the grid function and therefore the sequence itself converges toward the limiting function.

§ 4. The Wave Equation in Three Variables

We now consider the wave equation

$$2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{11}$$

and extend the remarks made in § 2 in connection with the region of dependence. The region of dependence of the differential equation (11) is the circular cone with its axis parallel to the t-direction and vertex angle α, with $\tan\alpha = \frac{1}{\sqrt{2}}$. In a rectangular grid we accordingly apply the difference equation

$$2u_{t\bar{t}} - u_{x\bar{x}} - u_{y\bar{y}} = 0. \tag{12}$$

Through this equation the function values u are connected with each other by the points of an "elementary octahedron". It allows the function value at a point S to be uniquely expressed by the function values at certain points of the two initial planes $t = 0$ and $t = h$. We obtain for each point S a determination pyramid which cuts two rhombi from the two base planes as a domain of dependence.

If we let the mesh width tend toward zero with retention of its proportions we can then expect convergence of the sequence of grid functions toward the solution of the differential equation only when the determination pyramid contains the determination cone of the differential equation in its interior. The simplest grid with this property will be the one which is placed such that the determination pyramid touches the determination cone from the outside. Our differential equation is chosen such that this occurs for a cubic rectangular grid.

In this grid the difference equation (12) assumes the following form in the notations of figure 10:

$$L(u) = \frac{2}{h^2}(u_\alpha' - 2u_0 + u_\alpha) - \frac{1}{h^2}(u_1 - 2u_0 + u_3) - \frac{1}{h^2}(u_2 - 2u_0 + u_4); \qquad (13)$$

here, moreover, the function value u_0 at the middle point P no longer enters. The values of the solution on the two initial planes are the values of a continuous function, four-times differentiable with respect to x_1, y_2, t.

For the proof of convergence we again use the method developed in § 3, forming for the solution of our difference equation the triple sum

$$h^3 \sum\sum\sum 2\,\frac{u_\alpha' - u_\alpha}{h}\,L(u) = 0$$

which extends over all octahedral elements of the determination pyramid emanating from a point S. We recognize on the basis of an almost literal interpretation of the previous conclusion that the values of the

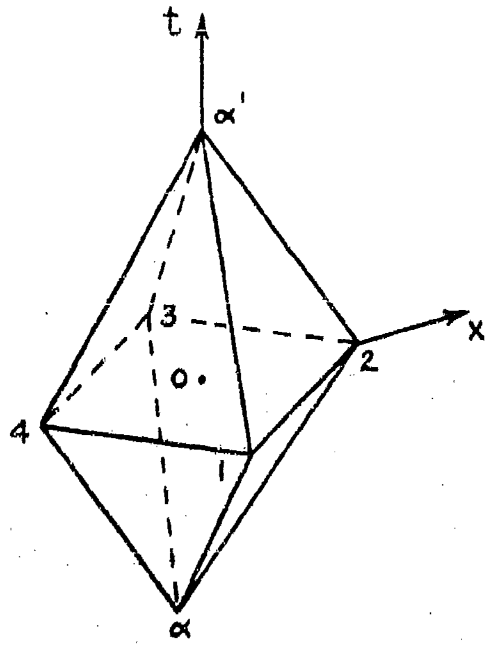

Fig. 10

function u at the interior points of the determination pyramid drop out, and that only surface sums remain over the four lateral double surfaces F and over the two initial planes I, II of the pyramid.

Denoting by u' the difference between function values at two points which are connected by an edge of an octahedral element, there results the formula

$$\sum_F \sum (u')^2 - \sum_{\mathrm{I}} \sum_{\mathrm{II}} (u')^2 = 0, \tag{14}$$

which is to be summed over all planes containing the differences u', so that each such difference appears only once.[26] Since the double sum

26 The grid is chosen such that the difference of u between the two planes F no longer occurs.

remains bounded over the two initial planes, as it transforms into an integral over initial values, the sum therefore also remains bounded over the "determination planes" F.

Instead of to u itself, let us turn our attention again to the first, second and third difference quotients which satisfy the difference equation (13) and their initial values; the initial values are themselves expressed by means of (13) by the first through fourth difference quotients formed from the values on the first two initial planes. If $w = w_h$ is one of these difference quotients up to the third order, we then know that the sum $h^2 \sum\sum \left(\frac{w'}{h}\right)^2$ remains bounded over one determination plane F. By exactly the same consideration which we applied in the first part of § 4, it follows that the function u and its first and second difference quotients are equi-continuous. Therefore there exists a sequence of mesh widths which decreases toward zero such that those expressions, which are bounded in the beginning, converge toward continuous limiting functions, and indeed obviously converge toward the solution of the differential equation including its first and second derivatives, which follows exactly as in § 3.

APPENDIX

SUPPLEMENTS AND GENERALIZATIONS

§ 5. Example of a Differential Equation of the First Order

We have seen in § 2 that under some circumstances the domain of dependence of the differential equation constitutes only a part of the domain of dependence of the difference equation, and therefore the influence of the remaining region drops out in the limit. We can explicitly demonstrate this phenomenon with the example of the first order differential equation $\partial u/\partial t = 0$ if we substitute for it the difference equation

$$2u_t - u_x + u_{\bar{x}} = 0. \tag{15}$$

Written in the notations of fig. 5 (p. 41) it reads

$$u_1 = \frac{u_2 + u_4}{2}. \tag{16}$$

This difference equation again connects only the points of a partial grid with one another. The initial value problem consists of specifying the function u as the values f_{2i} which a continuous function $f(x)$ assumes at the points $x = 2ih$ on the row $t = 0$.

Consider the point S on the t-axis at a distance $2nh$. It is easy to verify the representation of the solution u at S as

$$u_S = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \binom{2n}{n+i} f_{2i} \quad . \tag{17}$$

The sum on the right side tends toward the value f_0 with refinement of the mesh width, i.e., as $n \to \infty$. One can conclude this from the continuity of $f(x)$ and the behavior of the binomial coefficients with increasing n. (See the following paragraph.)

§ 6. The Equation of Heat Conduction

The difference equation (16) of § 5 may also be considered an analogue of a quite different equation, namely the equation of heat conduction

$$2\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 . \qquad (18)$$

In any rectangular grid the corresponding difference equation reads

$$2\left(\frac{u_1 - u_0}{l}\right) = \left(\frac{u_2 + u_4 - 2u_0}{h^2}\right) , \qquad (19)$$

where l is the time mesh and h the space mesh. In the limit of vanishing mesh width the difference equation maintains its form only when l decreases in proportion to h^2. In particular, if we set $l = h^2$, the value u_0 then drops out of the equation and there results the difference equation

$$u_1 = \frac{u_2 + u_4}{2}, \qquad (16)$$

whose solution has been given by formula (17):

$$u(0,t) = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \binom{2n}{n+i} f_{2i} . \qquad (17)$$

A point ξ of the x-axis is always denoted with decreasing mesh width by the index

$$2i = \frac{\xi}{h} . \qquad (20)$$

The mesh width is related to the ordinate t of the source point by the equation

$$2nh^2 = t . \qquad (21)$$

We shall now examine what results from formula (17) when h tends toward zero, i.e., as n tends toward infinity. By application of formula (21) we may write equation (17) in the form

$$u(0,t) = \sum_{i=-n}^{n} \frac{\sqrt{2n}}{2\cdot 2^{2n}\sqrt{t}} \binom{2n}{n+i} f_{2i} \cdot 2h . \qquad (22)$$

For the coefficients of $2hf_{2i} = 2hf(\xi)$ we use the abbreviation

$$\frac{1}{2\sqrt{t}}\, g_{2n}(\xi) = \frac{\sqrt{2n}}{2\cdot 2^{2n}\sqrt{t}} \binom{2n}{n+\frac{\xi}{\sqrt{2t}}\sqrt{n}} .$$

Here we shall calculate the limit of this coefficient, which one usually determines with the aid of Stirling's formula, by interpreting the function $g_{2n}(\xi)$ as a solution of an ordinary difference equation, and proceeding to the limit for vanishing mesh width h and thus to the differential equation. One finds

$$\frac{1}{2h}\left(g_h(\xi+2h) - g_h(\xi) \right) = -\frac{1}{2h}\, g_h(\xi)\, \frac{2i+1}{n+i+1}$$

as the difference equation by writing $g_h(\xi)$ instead of $g_{2n}(\xi)$. Or

$$\frac{1}{2h}\left(g_h(\xi+2h) - g_h(\xi) \right) = -g_h(\xi)\, \frac{\xi+h}{t+h\xi+2h^2} .$$

In addition $g_h(\xi)$ satisfies the normalization condition

$$\sum_{i=-n}^{n} g_h(\xi)\cdot 2h = 2\sqrt{t} .$$

This sum is to extend over the domain of dependance of the difference equation, which in the limit as $h \to 0$ occupies the whole x-axis.

One can see by simple considerations that $g_h(\xi)$ converges uniformly toward the solution $g(x)$ of the differential equation

$$g'(x) = -g(x)\,\frac{x}{t} ,$$

with the auxiliary condition

$$\int_{-\infty}^{\infty} g(x)\,dx = 2\sqrt{t} .$$

By proceeding to the limit there then arises from formula (22)

$$u(0,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\xi^2/2t} f(\xi)\, d\xi ,$$

the well-known solution of the equation of heat conduction.

The considerations of this paragraph are directly applicable to the case of the differential equation

$$4\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 ,$$

etc. in more independent variables.

§ 7. The General Linear Homogenous Differential Equation of Second Order in the Plane

Let us consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - k^2\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \gamma u = 0 . \qquad (23)$$

The coefficients are twice continuously differentiable with respect to x, t, while the initial values on the straight line $t = 0$ are three times continuously differentiable with respect to x. We replace the differential equation by the difference equation

$$L(u) = u_{t\bar{t}}(x,t) - k^2 u_{x\bar{x}}(x,t) + \alpha u_t + \beta u_x + \gamma u = 0 \qquad (24)$$

in a grid with the time mesh width h and the space mesh width $\varkappa h$, such that in the neighborhood of that part of the initial line under consideration $1 - \frac{h^2}{\varkappa^2} > \epsilon > 0$ holds for our constant $\varkappa$; we choose the initial values as in § 3. (See p. 49.)

For the convergence proof we again transform the sum

$$h^2 \sum_{S\alpha\beta}\sum 2\,\frac{u_1 - u_3}{h}\, L(u)$$

by application of the identities (7), (8). In addition to the sum (see (10)) over the boundary of the triangle $S\alpha\beta$ (see fig. 6) there now appears a sum over the whole triangle $S\alpha\beta$, whose absolute value can be estimated with the help of the Schwarz inequality as

$$C h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\grave{u}}{h}\right)^2 + \left(\frac{\acute{u}}{h}\right)^2 + \left(\frac{\dot{u}}{h}\right)^2 + u^2 \right] ,$$

where the constant C does not depend on the function u, on the mesh width h, or on the point S in a certain neighborhood of the initial line.

Here we can similarly estimate the value of $h^2 \sum_{S\alpha\beta}\sum u^2$ as

$$C_1 h^2 \sum_{S\alpha\beta}\sum \left(\frac{\dot{u}}{h}\right)^2 + C_2 h \sum_{\mathrm{I\,II}} u^2 ,$$

where what has been said for C will be valid for the constants C_1, C_2.[27]

We thus obtain an inequality of the form

$$\begin{aligned} h \sum_{S\alpha} \left[2\left(1 - \frac{k^2}{\kappa^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{\kappa^2}\left(\frac{\acute{u}}{h}\right)^2 \right] & \\ + h \sum_{S\beta} \left[2\left(1 - \frac{k^2}{\kappa^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{\kappa^2}\left(\frac{\grave{u}}{h}\right)^2 \right] & \\ \leq C_3 h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{\acute{u}}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right] + D , & \qquad (25) \end{aligned}$$

where D is a fixed constant for all the sums over the initial straight line, for all points S and mesh widths h.

Starting from the initial line we now choose as the point S of our triangle the points $S_0, S_1, \ldots, S_n = S$ in turn on a line parallel to the t-axis. By summation of the inequalities corresponding

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27 For the proof, see the related inequality at the bottom of p. 49.

to (25) we obtain the inequality

$$h^2 \sum_{S S_0 \alpha}\sum \left[2\left(1-\frac{k^2}{K^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{K^2}\left(\frac{\acute{u}}{h}\right)^2 \right]$$

$$+ h^2 \sum_{S S_0 \beta}\sum \left[2\left(1-\frac{k^2}{K^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{K^2}\left(\frac{\grave{u}}{h}\right)^2 \right]$$

$$\leq nh C_3 \sum_{S\alpha\beta}\sum \left[\left(\frac{\mathring{u}}{h}\right)^2 + \left(\frac{\acute{u}}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right] + nhD. \qquad (26)$$

Recalling that one can express the difference $\acute{u}$ or $\grave{u}$ in terms of the two differences $\dot{u}$ and a difference $\grave{u}$ or $\acute{u}$, respectively, it follows that we can then reduce the left side of (26) if we substitute for it

$$C_4 h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{\acute{u}}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right].$$

We now restrict ourselves to a neighborhood $t \leq nh$ of the initial line in which

$$C_4 - nhC_3 = C_5 > 0,$$

so we obtain from (26)

$$C_5 h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{\acute{u}}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right] \leq \frac{C_4}{C_5} D. \qquad (27)$$

From this expression of the boundedness of the left side of (27), it follows from (25) that

$$h \sum_{S\alpha} \left(\frac{\acute{u}}{h}\right)^2 + h \sum_{S\beta} \left(\frac{\grave{u}}{h}\right)^2$$

is also bounded, from which there follows the equi-continuity of u as in § 3.

Instead of applying the inequality (25) to the function u itself we apply it to the first and second difference quotients w, which also satisfy difference equations whose second-order terms are as in (24). In the additional terms derivatives of u which cannot be expressed by can indeed still occur, but the sum of their squares over a rectangular area multiplied by h^2, can already be assumed to be bounded. But in this case we may apply to this difference equation for w the same conclusions which have been applied earlier to u. Therefore we can infer the equi-continuity and boundedness of the functions u and their first and second derivatives, which therefore possess a partial sequence which converges uniformly toward the solution of the initial value problem of the differential equation. From the uniqueness it again follows that the function sequence itself converges.

In this case of course we must assume that the difference quotients up to the third order converge uniformly toward continuous limiting functions[28] on and between the initial rows.

§ 8. The Initial Value Problem of an Arbitrary Hyperbolic Linear Differential Equation of Second Order

We shall now show that the methods developed above are adequate to solve the initial value problem of an arbitrary linear homogenous hyperbolic differential equation of second order. In this case it is sufficient to restrict ourselves to the case of three variables. The train of thought may be immediately applied to several variables. One can easily see that the most general problem of this kind can be reduced

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[28] This assumption, and those concerning the differentiability of the coefficients of the differential equation and the boundedness in a sufficiently small neighborhood of the initial lines, can be relaxed in particular cases.

by a transformation of variables to the following: to find a function $u(x,y,t)$ which satisfies the differential equation

$$u_{tt} - (a u_{xx} + b u_{xy} + c u_{yy}) + \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0 \qquad (28)$$

and which, with its first derivatives, assumes prescribed values on the plane $t=0$. In this case the coefficients may be functions of the variables x, y, t and should satisfy the conditions

$$a > 0\,, \quad c > 0\,, \quad ac - b^2 > 0\,.$$

We also assume that the coefficients are three times continuously differentiable with respect to x, y, t, and that the initial values u are four times differentiable and u_t three times differentiable with respect to x, y.

We now assume the coordinates x and y are rotated around a point in the initial plane such that $b=0$. Then in a certain neighborhood G of this point the conditions

$$a - |b| > 0\,, \quad c - |b| > 0$$

are surely fulfilled. We limit our considerations to this neighborhood. We then can choose a three times continuously differentiable function $d > 0$ such that

$$\left.\begin{matrix} a-d \\ c-d \\ d-|b| \end{matrix}\right\} > \varepsilon > 0 \qquad (29)$$

is valid with constant ε. We can then put the differential equation in the form

$$u_{tt} - (a-d)u_{xx} - (c-d)u_{yy} - \tfrac{1}{2}(d+b)(u_{xx} + 2u_{xy} + u_{yy})$$
$$-\tfrac{1}{2}(d-b)(u_{xx} - 2u_{xy} + u_{yy}) + \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0. \qquad (30)$$

We now construct the grid of points

$$t = lh, \quad x+y = m\varkappa h, \quad x-y = n\varkappa h \quad (l, m, n = \cdots, -1, 0, 1, 2, \cdots)$$

in the region and substitute for equation (30) a difference equation $L(u) = 0$ in this grid. For this purpose we adjoin to each grid point P_0 the following adjacent points: The points P_α' and P_α, which come from P_0 by displacements h and $-h$, respectively, in the direction of the t-axis; and the points $P_1, \ldots, P_8$ which lie in the same plane as P_0 parallel to the (x, y)-plane; see fig. 11. These points form an "octahedral element" with the corner points P_α', P_α, P_1, P_2, P_3, P_4.

For each grid point P_0 which is inside G we replace the second differential quotients occuring in (30) by difference quotients

Fig. 11

about P_0 in the octahedral element in the following way.

We replace

$$u_{tt} \quad \text{by} \quad \frac{1}{h^2}(u'_\alpha - 2u_0 + u_\alpha),$$

$$u_{xx} \quad \text{by} \quad \frac{1}{\kappa^2 h^2}(u_2 - 2u_0 + u_4),$$

$$u_{yy} \quad \text{by} \quad \frac{1}{\kappa^2 h^2}(u_1 - 2u_0 + u_3),$$

$$u_{xx} + 2u_{xy} + u_{yy} \quad \text{by} \quad \frac{4}{\kappa^2 h^2}(u_6 - 2u_0 + u_8),$$

$$u_{xx} - 2u_{xy} + u_{yy} \quad \text{by} \quad \frac{4}{\kappa^2 h^2}(u_5 - 2u_0 + u_7).$$

We substitute corresponding difference quotients in the octahedral element for the first differential quotients occurring in (30). We give the coefficients in the difference equation the values which are taken by the coefficients of the differential equation at the point P_0.

On the first two initial planes $t=0$ and $t=h$ we assume the function values are prescribed so that they converge toward the prescribed initial values at $t=0$ with refinement of the mesh width while holding the ratio κ of the space- to the time-mesh width fixed, whereby the difference quotients up to the fourth order formed between the two planes $t=0$ and $t=h$ should uniformly converge toward the corresponding prescribed differential quotients.

The solution of the difference equation $L(u)=0$ at a point is uniquely determined by the values on the two bottom planes of the determination pyramid passing through it.

For the proof of convergence we form over all elementary octahedrons of a determination pyramid the sum

$$h^3 \sum\sum\sum 2\,\frac{u_\alpha' - u_\alpha}{h}\,L(u)$$

and transform it by means of the identities (7), (8). In this way there arises one spatial sum multiplied by h^3, which is quadratic in the first difference quotients, and also a sum multiplied by h^2 over the lateral double planes, in which appear the squares of <u>all</u> difference quotients of the type $u_\alpha - u_0$, $u_\alpha - u_1$, ..., $u_\alpha - u_8$ occurring on and between the double planes; their coefficients are greater than a fixed positive constant because of (29), if we choose moreover a sufficiently small ratio $\frac{1}{k}$ of time- to space-mesh widths.

From here we can proceed in the same fashion as in §§ 7, 4, and can prove that the solution of our difference equation converges toward the solution of the differential equation.

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